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On the geometry of commuting polarities

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Abstract

Some geometry and combinatorics of orthogonal and symplectic polarities commuting with a unitary polarity are investigated.

0. Introduction

The notion of commuting polarities was introduced by Tits in 1955 [13]. In a paper completely devoted to the Hermitian geometry over a finite field [8], Segre developed the theory of polarities commuting with a unitary polarity. We believe that Segre's paper represents one of the most beautiful chapters in finite geometries.

In this paper we investigate some geometry and combinatorics of orthogonal and symplectic polarities commuting with unitary polarities.

In Section 1, after some basic preliminaries on Hermitian curves and Hermitian surfaces, we introduce the geometry of symplectic and orthogonal polarities commuting with a unitary polarity.

In Section 2, we report some known facts about conics commuting with a Hermitian curve and in particular show that such conics may always be used as building blocks for at least classical unitals.

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In Section 3, the starting point is the construction of the rational curve $\mathcal{K} := \{(1, t, t^{q+1} \mid t \in GF(q^2))\} \cup \{(0, 0, 1)\}$ as the union of all $q+1$ conics of $PG(2, q^2)$, q odd, commuting with the Hermitian curve of equation $X_0X_2^q - 2X_1^{q+1} + X_2X_0^q = 0$ and passing through the two points $(1, 0, 0)$ and $(0, 0, 1)$. The point set of the curve $\mathcal{K}$ is a set of type $[0, 1, 2, q+1]_1$ (i.e., it is a subset of points of $PG(2, q^2)$ such that any line of $PG(2, q^2)$ meets it in either 0, 1, 2 or $q+1$ points). Such a set is called a C_F-set (after [3]). Next, we investigate some connections between C_F-sets and elliptic quadrics of $PG(3, q)$. We conclude this section using again some geometry of orthogonal polarities commuting with a unitary polarity in order to construct a new subset of type $[0, 1, 2, q+1]_1$ of a Hermitian curve different from the C_F-set mentioned above.

In the last section, first and most important, we show that there exists a connection between the curve $\mathcal{K}$ and regular symplectic spreads of $PG(3, q)$.

Some effort was made to ensure a coherent and well-harmonized presentation of the various sections.

1. Preliminaries

Let $PG(2, q^2)$ denote the Desarguesian projective plane over the finite field $GF(q^2)$, where q is any prime power. A *Hermitian curve* $\mathcal{H}(2, q^2)$ in $PG(2, q^2)$ is defined as the set of all isotropic points of a non-degenerate unitary polarity $\perp$ of $PG(2, q^2)$. The number of points of $\mathcal{H}(2, q^2)$ is $q^3 + 1$. If P is a point in $PG(2, q^2)$, then the polar line $P^\perp$ of P with respect to $\mathcal{H}(2, q^2)$ meets $\mathcal{H}(2, q^2)$ in 1 or $q+1$ points, according as whether P lies on $\mathcal{H}(2, q^2)$ or does not. Similarly, if l is a line in $PG(2, q^2)$, the pole $l^\perp$ of l with respect to $\mathcal{H}(2, q^2)$ lies on $\mathcal{H}(2, q^2)$ or does not, according as whether l meets $\mathcal{H}(2, q^2)$ in 1 or $q+1$ points. Lines of the first type are called *tangents*, while those of the second type are called *secants* or *chords* of $\mathcal{H}(2, q^2)$. There is just one tangent at every point $P \in \mathcal{H}(2, q^2)$, whereas the remaining q^2 lines through P are secants. If $P \notin \mathcal{H}(2, q^2)$, then through P there are $q+1$ tangents (meeting $\mathcal{H}(2, q^2)$ in the points of $P^\perp \cap \mathcal{H}(2, q^2)$) and $q^2 - q$ secants.

Let $PGU_3(q^2)$ denote the projective unitary group associated with $\perp$, then the following properties hold (for more details see [7] and [11]).

(a) The group $PGU_3(q^2)$ acts doubly transitively on the set of points of $\mathcal{H}(2, q^2)$.
(b) If P is a non-isotropic point in $PG(2, q^2)$, then the group of all unitary homologies with centre P and axis the polar line $P^\perp$ of P is a cyclic group of order $q+1$.
(c) If P is a point of $\mathcal{H}(2, q^2)$, then the group of all unitary elations with centre P and axis $P^\perp$ has order q.
(d) If P and Q are two distinct points of $\mathcal{H}(2, q^2)$, then the stabilizer in $PGU_3(q^2)$ of the two points P and Q is a cyclic group of order $q^2 - 1$.
(e) If P is a point of $\mathcal{H}(2, q^2)$ then the stabilizer H in $PGU_3(q^2)$ of P contains a normal subgroup of order q^3 which acts transitively on the set of points of $\mathcal{H}(2, q^2)$ distinct from P. The centre of H coincides with the commutator subgroup which turns out to be the group of all unitary elations with centre P and axis the polar line $P^\perp$ of P.

In $PG(3, q^2)$ a *Hermitian surface* is defined to be the set of all isotropic points of a non-degenerate unitary polarity, and it is denoted by $\mathcal{H}(3, q^2)$.

A Hermitian surface $\mathcal{H}(3, q^2)$ has the following properties (for more details, see [8]).

1. The number of points on it is $(q^2 + 1)(q^3 + 1)$.
2. Any line of $PG(3, q^2)$ meets $\mathcal{H}(3, q^2)$ in either 1 or $q+1$ or $q^2 + 1$ points. The latter lines are the *generators* of $\mathcal{H}(3, q^2)$, and they are $(q+1)(q^3 + 1)$ in number. The intersections of

size $q+1$ are Baer sublines, whereas lines meeting $\mathcal{H}(3,q^2)$ in one point are called *tangent lines*.

3. Through every point P of $\mathcal{H}(3,q^2)$ there pass exactly $q+1$ generators, and these generators are coplanar. The plane containing these generators, say π_P, is the polar plane of P with respect to the unitary polarity defining $\mathcal{H}(3,q^2)$. The tangent lines through P are precisely the remaining q^2-q lines of π_P incident with P, and π_P is called the *tangent plane* to $\mathcal{H}(3,q^2)$ at P.

   A *t-span* of $\mathcal{H}(3,q^2)$ is a set of t pairwise disjoint generators, and it is said to be *complete* if it is not contained in a $(t+1)$-span.
4. Every plane of $PG(3,q^2)$ which is not a tangent plane to $\mathcal{H}(3,q^2)$ is called a *secant plane* and meets $\mathcal{H}(3,q^2)$ in a *Hermitian curve* $\mathcal{H}(2,q^2)$.

1.1. *Commuting unitary and orthogonal polarities*

Let $\mathcal{H}(3,q^2)$ be a Hermitian surface of $PG(3,q^2)$. Let $\mathcal{B}$ be an orthogonal polarity commuting with the unitary polarity $\mathcal{U}$ associated with $\mathcal{H}(3,q^2)$. Set $\mathcal{V}=\mathcal{B}\mathcal{U}=\mathcal{U}\mathcal{B}$. Then (see [8]) $\mathcal{V}$ is a non-linear collineation and fixes either $(q+1)^2$ or q^2+1 points on $\mathcal{H}(3,q^2)$, yielding a hyperbolic quadric $Q^+(3,q)$ or an elliptic quadric $Q^-(3,q)$, respectively, embedded in a Baer subgeometry B of $PG(3,q^2)$. In both cases $B\cap\mathcal{H}(3,q^2)=Q^{\pm}(3,q)$. Notice that the points of $\mathcal{H}(3,q^2)$ fixed under $\mathcal{V}$ are those admitting the same tangent plane with respect to both the orthogonal polarity and the unitary polarity. Hence, the projective orthogonal group $PGO_4^{\epsilon}(q)$, $\epsilon=\pm$, associated with $\mathcal{B}$ turns out to be a subgroup of the projective unitary group $PGU_4(q^2)$ associated with $\mathcal{U}$.

In terms of forms, let us assume that (V,g) is a 4-dimensional unitary space over $GF(q^2)$. Let F be the subfield of $GF(q^2)$ of index two, i.e., $GF(q)$. Choose a basis $\mathbf{b}=\{v_1,\ldots,v_4\}$ of V such that $g(v_i,v_j)\in F$ for all i and j, and let W denote the F-span of $\mathbf{b}$. The restriction $\bar{g}$ of g to W is a non-degenerate symmetric bilinear form. If $\mathbf{b}$ is an orthonormal basis, then the discriminant of $\bar{g}$ is a square. By replacing v_1 by βv_1, where β is a generator of $GF(q^2)^*$, the discriminant of $\bar{g}$ becomes a non-square. Therefore, we obtain the embeddings $O_4^{\epsilon}(q)<GU_4(q^2)$, for both $\epsilon=+$ and $\epsilon=-$. Factoring out scalars, we get the embeddings $PO_4^{\epsilon}(q)<PGU_4(q^2)$.

1.2. *Commuting unitary and symplectic polarities*

Let $\mathcal{A}$ be a symplectic polarity commuting with the unitary polarity $\mathcal{U}$ associated with $\mathcal{H}(3,q^2)$. Set $\mathcal{V}=\mathcal{A}\mathcal{U}=\mathcal{U}\mathcal{A}$. Then (see [8]) $\mathcal{V}$ is a non-linear collineation, it fixes q^3+q^2+q+1 points on $\mathcal{H}(3,q^2)$, but fixes no point outside $\mathcal{H}(3,q^2)$, and leaves invariant the same number of generators of $\mathcal{H}(3,q^2)$. Each fixed point is incident with $q+1$ invariant generators, and each invariant generator is incident with $q+1$ fixed points. This symmetric configuration $\mathcal{W}$ on $\mathcal{H}(3,q^2)$ extends to a 3-dimensional projective space $\Sigma_0\cong PG(3,q)$, by adding the $q^2(q^2+1)\mathcal{V}$-invariant lines which are not generators of $\mathcal{H}(3,q^2)$. In this context, Σ_0 is naturally equipped with the symplectic polarity $\mathcal{A}$ whose isotropic lines are the lines of the above symmetric configuration $\mathcal{W}$, and $\mathcal{W}\cong\mathcal{W}_3(q)$ is a symplectic polar space. If $\mathcal{H}(3,q^2)$ has canonical equation $X_0^{q+1}+X_1^{q+1}+X_2^{q+1}+X_3^{q+1}=0$, then $\mathcal{W}$ can be described as the subset of points of $\mathcal{H}(3,q^2)$ whose coordinates are of the form $(a,\rho a^q,b,\rho b^q)$, where $a,b,\rho\in GF(q^2)$ with $\rho^{q+1}=-1$ (see [8]). The projective symplectic group $PSp_4(q)$ associated with $\mathcal{A}$ turns out to be a subgroup of the projective unitary group $PGU_4(q^2)$ associated with $\mathcal{U}$.

In terms of forms, we start from a symplectic geometry (V, A), where V is a 4-dimensional vector space over $GF(q)$ and A is a symplectic form on V. Let ω be an element of $GF(q^2) \setminus GF(q)$. Then $GF(q^2) = GF(q) \oplus GF(q)\omega$, and we define $W := \{(\alpha + \beta\omega)v \mid \alpha, \beta \in GF(q), v \in V\} \cong V \otimes GF(q^2)$. Any vector $w \in W$ can be written as $w = \sum v_i \otimes (a_i + b_i\omega) = \sum(v_i \otimes 1)a_i + \sum(v_i \otimes \omega)b_i = (\sum v_i a_i) \otimes 1 + (\sum v_i b_i) \otimes \omega$, which we rewrite as $w_1 + w_2\omega$. Using this notation, we have $(\alpha + \beta\omega)(w_1 + w_2\omega) = (\alpha w_1 + \beta\omega^2 w_2) + (\beta w_1 + \alpha w_2)\omega$. There is a natural extension of A to an anti-Hermitian form $C : W \times W \to GF(q^2)$ given by:

$$C(w_1 + w_2\omega, v_1 + v_2\omega) = A(w_1, v_1) + \omega\omega^q A(w_2, v_2) + \omega A(w_2, v_1) + \omega^q A(w_1, v_2).$$

If q is even, then C is already a Hermitian form. In all cases, there exists $\tau \in GF(q^2)$ such that $\tau^q = -\tau$ and τC is a Hermitian form with the same group as C. We write h for τC and note that h does not depend on the choice of ω. Denoting by $GU_4(q^2)$ the group of all similarities of h and by $Sp_4(q)$ the group of all isometries of A, we obtain the embedding $Sp_4(q) \leq GU_4(q^2)$. Factoring out scalars yields the embedding $PSp_4(q) \leq PGU_4(q^2)$.

2. Some geometry of commuting orthogonal and unitary polarities

In this section we investigate some geometry of commuting orthogonal and unitary polarities. Fix any two points on a Hermitian curve $\mathcal{H}(2, q^2)$, say P and Q, then the number of conics of $PG(2, q^2)$, q odd, commuting with $\mathcal{H}(2, q^2)$ and passing through P and Q is $q + 1$. Each of these conics intersects $\mathcal{H}(2, q^2)$ in a Baer subconic the union of which turns out to be a rational curve $\mathcal{X}$. The curve $\mathcal{X}$ has many nice properties and its geometry is related to other combinatorial structures.

2.1. Conics commuting with a Hermitian curve

Let $\mathcal{H}(2, q^2)$ denote a Hermitian curve, $q = p^h$, with p an odd prime and $h \geq 1$. There are $q^2(q^3 + 1)$ conics of $PG(2, q^2)$ commuting with $\mathcal{H}(2, q^2)$ on which the group $PGU_3(q^2)$ acts transitively. Each of the above conics meets $\mathcal{H}(2, q^2)$ in a subconic of a Baer subplane $PG(2, q)$ of $PG(2, q^2)$. It follows that the number of these subconics is $q^2(q^3 + 1)$ as well, and each of them contains $\frac{q(q+1)}{2}$ pairs of distinct points of $\mathcal{H}(2, q^2)$. Since $\mathcal{H}(2, q^2)$ has $q^3 + 1$ points and $PGU_3(q^2)$ acts 2-transitively on them, we conclude that through any two distinct points of $\mathcal{H}(2, q^2)$ there pass $q + 1$ conics commuting with $\mathcal{H}(2, q^2)$. Through any three distinct points of $\mathcal{H}(2, q^2)$ exactly one conic passes commuting with $\mathcal{H}(2, q^2)$ [8, Section 53]. For more details on conics commuting with a Hermitian curve, and some related interesting properties, see [8] (see also [4]).

Now, we define the incidence structure U as follows. The points of U are those of $\mathcal{H}(2, q^2)$ and the blocks of U are the subconics. It turns out that U is a 3-$(q^3 + 1, q + 1, 1)$-design and the number of blocks on a point, that is, the number of conics of $PG(2, q^2)$ commuting with $\mathcal{H}(2, q^2)$ and passing through a point is $q^2(q + 1)$. Let $\mathcal{C}$ be one of these subconics and P a point of $\mathcal{C}$. The stabilizer of P in $PGU_3(q^2)$ contains a p-subgroup H of order q^3, fixing the point P and acting transitively on the remaining points of $\mathcal{H}(2, q^2)$. The centre of H coincides with the commutator subgroup H' and has order q. Since H is transitive on the points of $\mathcal{H}(2, q^2) \setminus \{P\}$, for every point Q of $\mathcal{H}(2, q^2)$ different from P there is a subconic through P and Q. Let J be a subgroup of H of order q^2 such that the quotient J/H' is not isomorphic to H'. Then, J is an elementary abelian group that does not fix $\mathcal{C}$ and so the orbit of $\mathcal{C}$ under J gives q^2 subconics, any two of them sharing only the point P. These q^2 copies of $\mathcal{C}$ cover the Hermitian curve $\mathcal{H}(2, q^2)$.

Let $\mathcal{H}(2, q^2)$ be the Hermitian curve of $PG(2, q^2)$, q odd, with equation

$$X_0 X_2^q - 2X_1^{q+1} + X_2 X_0^q = 0.$$

Consider the family $\mathcal{C} = \{\mathcal{C}_a\}$ of non-singular conics of $PG(2, q^2)$

$$\mathcal{C}_a : X_0 X_2 - a X_1^2 = 0,$$

where a is a $(q+1)$-st root of unity in $GF(q^2)$. It is easily seen that $\mathcal{C}_a$ commutes with $\mathcal{H}(2, q^2)$ (see [8]), and any two conics of the family $\mathcal{C}$ share only the two points $P_0 := (1, 0, 0)$ and $P_\infty := (0, 0, 1)$. Since $|\mathcal{C}| = q + 1$, $\mathcal{C}$ is the set of all conics of $PG(2, q^2)$ commuting with $\mathcal{H}(2, q^2)$ and passing through P_0 and P_∞.

Set $\mathcal{Q}_a := \mathcal{C}_a \cap \mathcal{H}(2, q^2)$. From [8], $\mathcal{Q}_a$ is a conic defined over $GF(q)$. In particular, a point which lies on $\mathcal{C}_a$ has homogeneous projective coordinates $(0, 0, 1)$ or $(1, t, at^2)$, with $t \in GF(q^2)$. Obviously, $P_\infty \in \mathcal{H}(2, q^2)$, hence $P_\infty \in \mathcal{Q}_a$. The point $P_t := (1, t, at^2)$ lies on the Hermitian curve $\mathcal{H}(2, q^2)$ if and only if $t^2(a^q t^{2q-2} - 2t^{q-1} + a) = 0$. For $t = 0$, we get the point $P_0 = (1, 0, 0)$. If $t \neq 0$, setting $s = t^{q-1}$ yields $a^q s^2 - 2s + a = 0$. Then, $P_t \in \mathcal{Q}_a$, with $t \in GF(q^2)^*$, if and only if $t^{q-1} = a$. Since $a^{q+1} = 1$, this equation admits $q - 1$ distinct solutions. Therefore

$$\mathcal{Q}_a = \{(1, t, at^2) \mid t \in GF(q^2)^*, t^{q-1} = a\} \cup \{P_0, P_\infty\}.$$

Set $\mathcal{K} := \bigcup \mathcal{Q}_a$. Then, $\mathcal{K}$ has size $q^2 + 1$. It is easily seen that each point of $\mathcal{K}$ lies on the curve $\mathcal{X}$ with equation $X_0 X_2^q - X_1^{q+1} = 0$. Hence, the set

$$\mathcal{K} = \{(1, t, t^{q+1}) \mid t \in GF(q^2)\} \cup \{(0, 0, 1)\}$$

coincides with the set of $q^2 + 1 GF(q^2)$-rational points of $\mathcal{X}$. The set $\mathcal{K}$ is a set of type $[0, 1, 2, q + 1]_1$. Notice that the set $\mathcal{K}$ coincides with one of the sets studied by Donati and Durante in [3].

From now on we refer to the set $\mathcal{K}$ (for any q) as a C_F-set (after [3]).

2.2. C_F-sets and elliptic quadrics of $PG(3, q)$

In this section we wish to prove the following theorem.

Theorem 2.1. *To a C_F-set of $\mathcal{H}(2, q^2)$ there corresponds an elliptic quadric $\mathcal{Q}^-(3, q)$ of $PG(3, q)$, and conversely.*

Proof. Let X_0, X_1, X_2 be homogeneous projective coordinates in $PG(2, q^2)$, q a prime power. Consider the C_F-set $\mathcal{K} = \{(1, t, t^{q+1}) \mid t \in GF(q^2)\} \cup \{P_\infty = (0, 0, 1)\}$, and the map $\Phi : \mathcal{K} \to PG(3, q^2)$ which maps any point $P_t := (1, t, t^{q+1})$ of $\mathcal{K}$ to $\Phi(P_t) := (1, t, t^q, t^{q+1})$ and P_∞ to $\Phi(P_\infty) := (0, 0, 0, 1)$. The map Φ is injective. The set $\mathcal{Q} := \Phi(\mathcal{K}) = \{(1, t, t^q, t^{q+1}) \mid t \in GF(q^2)\} \cup \{(0, 0, 0, 1)\}$ corresponds to an elliptic quadric $\mathcal{Q}^-(3, q)$ of $PG(3, q)$.

Conversely, denote by X_0, X_1, X_2, X_3 homogeneous projective coordinates in $PG(3, q^2)$, and set $Q := (0, 0, 1, 0)$. The point Q does not lie on the elliptic quadric $\mathcal{Q}$. So, the projection ρ of $\mathcal{Q}$ from Q onto the projective plane $\pi : X_2 = 0$ of $PG(3, q^2)$ is well defined. Obviously, since $Q_0 := (1, 0, 0, 0)$ and $Q_\infty := (0, 0, 0, 1)$ lie on π, it follows that $\rho(Q_0) = Q_0$ and $\rho(Q_\infty) = Q_\infty$. For any $t \in GF(q^2)^*$, setting $Q_t := (1, t, t^q, t^{q+1})$, the line $Q_t Q$ corresponds to the point set $\{(1, t, t^q + \lambda, t^{q+1}) \mid \lambda \in GF(q^2)\} \cup \{(0, 0, 1, 0)\}$ and it turns out that

$\rho(Q_t) := Q_t Q \cap \pi = (1, t, 0, t^{q+1})$. Now, observe that there are no 2-secant lines of $\mathcal{Q}$ passing through Q. Indeed, if $Q_{t_1} := (1, t_1, t_1^q, t_1^{q+1})$ and $Q_{t_2} := (1, t_2, t_2^q, t_2^{q+1})$ are any two distinct points of $\mathcal{Q}$, with $t_1, t_2 \in GF(q^2) \cup \{\infty\}$ and $t_1 \neq t_2$, the point Q lies on the line $Q_1 Q_2 := \{(1 + \lambda, t_1 + \lambda t_2, t_1^q + \lambda t_2^q, t_1^{q+1} + \lambda t_2^{q+1}) \mid \lambda \in GF(q^2)\} \cup \{(1, t_2, t_2^q, t_2^{q+1})\}$ if and only if $\lambda = -1$, that is, $t_1 = t_2$, a contradiction. Therefore, $|\rho(\mathcal{Q})| = q^2 + 1$ and one can choose homogeneous projective coordinates in the plane π in such a way that $\rho(\mathcal{Q})$ is the set of points $\{(1, t, t^{q+1} \mid t \in GF(q^2))\} \cup \{(0, 0, 1)\}$. Such a set coincides with the C_F-set $\mathcal{K}$. $\square$

Remark 2.2. The Klein representation of the lines of $PG(3, q^2)$ as points of $\mathcal{Q}^+(5, q^2)$ yields that to the totally isotropic lines of a unitary polarity $\mathcal{U}$ of $PG(3, q^2)$ associated with a Hermitian surface $\mathcal{H}(3, q^2)$ correspond the points of an elliptic quadric $\mathcal{Q}^-(5, q)$ (see [1]). Let π be a plane of $PG(5, q^2)$ which is non-degenerate with respect to the orthogonal polarity $\perp$ of $PG(5, q^2)$ defined by the quadratic form associated with $\mathcal{Q}^+(5, q^2)$. The planes π and $\pi^\perp$ intersect $\mathcal{Q}^+(5, q^2)$ in a conic. Each point on one conic is orthogonal to every point on the second one. Thus, such conics correspond to the set of lines in $PG(3, q^2)$ which lie on a hyperbolic quadric $\mathcal{Q}^+(3, q^2)$. By [8] the intersection of $\mathcal{H}(3, q^2)$ and $\mathcal{Q}^+(3, q^2)$ lies in a subgeometry $PG(3, q)$ where it forms an elliptic quadric $\mathcal{Q}^-(3, q)$. The q^2+1 points of $\mathcal{Q}^-(3, q)$ correspond to the lines of a partial spread S_0 of $\mathcal{Q}^-(5, q)$ (see [2]) or, to be more precise, the $q + 1$ points on a line of S_0 correspond to the $q + 1$ tangent lines through a point of $\mathcal{Q}^-(3, q)$. Such partial spread S_0 of $\mathcal{Q}^-(5, q)$ is a so-called BLT-set (see [9]), that is, a set of $q^2 + 1$ mutually skew lines of $\mathcal{Q}^-(5, q)$ with the property that every line of $\mathcal{Q}^-(5, q)$, which is not a member of S_0, meets non-trivially exactly two or none of the lines of S_0. For more details see also [2].

Remark 2.3. From [3] it follows that there are $q - 1(q + 1)$-secant lines to $\mathcal{K}$ through the point $P := (0, 1, 0)$, which is not on $\mathcal{K}$. Firstly, observe that the line through P with equation $X_2 = 0$ meets the set $\mathcal{K}$ only at the point P_0. The line $X_0 + \lambda X_2 = 0$, $\lambda \in GF(q^2)$, is a $(q + 1)$-secant to $\mathcal{K}$ if and only if $\lambda = -\frac{1}{t^{q+1}}$, that is, if and only if $\lambda^{q-1} = 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}$ be the $(q - 1)$-st distinct roots of unity in $GF(q^2)$. Any line $r_{\lambda_i} : X_0 + \lambda_i X_2 = 0$, for $i = 1, \ldots, q - 1$, meets $\mathcal{K}$ in the point set $r_{\lambda_i} \cap \mathcal{K} = \{(1, t, t^{q+1}) \mid t \in GF(q^2)^*,\ t^{q+1} = -\frac{1}{\lambda_i}\}$. Since $\lambda_i^{q-1} = 1$, the equation $t^{q+1} = -\frac{1}{\lambda_i}$ admits $q + 1$ distinct solutions in $GF(q^2)$. Hence, $|r_{\lambda_i} \cap \mathcal{K}| = q + 1$, for any $i \in \{1, \ldots, q - 1\}$.

Remark 2.4. Using the same arguments as in Remark 2.3, if λ_i is a $(q - 1)$-st root of unity in $GF(q^2)$, then the point set $\{(1, t, t^{q+1}) \mid t \in GF(q^2)^*, t^{q+1} = -\frac{1}{\lambda_i}\}$ of $PG(2, q^2)$ corresponds to the $q + 1$ points of intersection between $\mathcal{K}$ and the $(q + 1)$-secant line r_{λ_i} : $X_0 + \lambda_i X_2 = 0$ to $\mathcal{K}$. The map Φ defined in this section takes such a point set to the point set $\mathcal{F}_{\lambda_i} := \{(1, t, t^q, t^{q+1}) \mid t \in GF(q^2)^*, t^{q+1} = -\frac{1}{\lambda_i}\}$ of $PG(3, q^2)$. Such a set $\mathcal{F}_{\lambda_i}$ has $q + 1$ points, and these points lie on $\mathcal{Q}$. The set $\mathcal{F}_{\lambda_i}$ is contained in the plane of $PG(3, q^2)$ with equation $X_0 + \lambda_i X_3 = 0$. Hence, $\mathcal{F}_{\lambda_i}$ is a non-singular conic defined over $GF(q)$ and is contained in $\mathcal{Q}$. In this way, we obtain a partition of the point set of $\mathcal{Q} \setminus \{Q_0, Q_\infty\}$ into $q - 1$ mutually disjoint conics defined over $GF(q)$ not passing through Q_0, Q_∞. Such conics form the so-called linear flock of the elliptic quadric $\mathcal{Q}$ (see [12]).

Remark 2.5. Now, fix the points $Q_0 := (1, 0, 0, 0)$ and $Q_\infty := (0, 0, 0, 1)$ on the elliptic quadric $\mathcal{Q}$. Any plane of $PG(3, q^2)$ through the line $Q_0 Q_\infty$ and distinct from the plane $\pi : X_2 = 0$, has equation $X_1 + \lambda X_2 = 0$, where $\lambda \in GF(q^2)$. A plane $\pi_\lambda : X_1 + \lambda X_2 = 0$ intersects $\mathcal{Q}$ in a conic defined over $GF(q)$ if and only if the equation $t + \lambda t^q = 0$ has $q + 1$ distinct solutions. This holds if and only if $\lambda^{q+1} = 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{q+1}$ be the $(q + 1)$-st distinct roots of unity in $GF(q^2)$. Any plane $\pi_{\lambda_i} : X_1 + \lambda_i X_2 = 0$, for $i = 1, \ldots, q + 1$, meets $\mathcal{Q}$ in the conic Γ_{λ_i} whose point set is the following

$$\left\{(1, t, t^q, t^{q+1}) \mid t \in GF(q^2)^*, t^{q-1} = -\frac{1}{\lambda_i}\right\} \cup \{Q_0, Q_\infty\}.$$

Using the projection ρ of $\mathcal{Q}$ from $Q := (0, 0, 1, 0)$ on the plane π and arguing as in the proof of Theorem 2.1, it can be proved that $\rho(\Gamma_{\lambda_i})$ corresponds to the point set of $PG(2, q^2)$

$$\left\{(1, t, t^{q+1}) \mid t \in GF(q^2)^*, t^{q-1} = -\frac{1}{\lambda_i}\right\} \cup \{P_0 = (1, 0, 0), P_\infty = (0, 0, 1)\}.$$

Since ρ is an injective map $|\rho(\Gamma_{\lambda_i})| = q + 1$. As $\lambda_i^{q+1} = 1$, for any $i \in \{1, \ldots, q + 1\}$, it is easily seen that the points of $\rho(\Gamma_{\lambda_i})$ lie on the non-singular conic $\mathcal{C}_{\lambda_i}$ of $PG(2, q^2)$ with equation $X_0X_2 + \frac{1}{\lambda_i}X_1^2 = 0$. By previous arguments, if q is odd, these conics are all conics of $PG(2, q^2)$ commuting with the Hermitian curve $\mathcal{H}(2, q^2)$ with equation $X_0X_2^q - 2X_1^{q+1} + X_2X_0^q = 0$ and passing through P_0 and P_∞. It turns out that $\rho(\Gamma_{\lambda_i}) = \mathcal{C}_{\lambda_i} \cap \mathcal{H}(2, q^2)$ and the point set $\bigcup_{i=1}^{q+1} \rho(\Gamma_{\lambda_i})$ coincides with the C_F-set $\mathcal{K}$.

Remark 2.6. Set $P := (0, 0, 1)$ and denote by H' the root group of $SU_3(q^2)$ with centre P and axis $P^\perp$. The group H' has order q and its elements are represented by matrices of the form

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in GF(q^2)$ and $\alpha^q + \alpha = 0$. Such a group acts on $\mathcal{K}$ fixing only the point P. The action of H' takes $\mathcal{K}$ to the point set

$$\{(1, t, t^{q+1} + \alpha) \mid t, \alpha \in GF(q^2),\ \alpha^q + \alpha = 0\} \cup \{(0, 0, 1)\},$$

which coincides with the Hermitian curve $\mathcal{H}(2, q^2)$.

Finally, we prove the following result.

Proposition 2.7. *The C_F-set $\mathcal{K}$ is the base locus of q Hermitian curves of $PG(2, q^2)$.*

Proof. Let $\mathcal{H}$ be an arbitrary Hermitian curve of $PG(2, q^2)$ with equation

$$aX_0^{q+1} + bX_1^{q+1} + cX_2^{q+1} + dX_0^qX_1 + d^qX_0X_1^q + eX_0^qX_2 + e^qX_0X_2^q + fX_1^qX_2 + f^qX_1X_2^q = 0.$$

Recall that $\mathcal{K} = \{(1, t, t^{q+1}) \mid t \in GF(q^2)\} \cup \{(0, 0, 1)\}$. If $\mathcal{K}$ is contained in $\mathcal{H}$, then $a = c = d = f = 0$ and $b + e + e^q = 0$. Thus, the equation of $\mathcal{H}$ becomes

$$eX_0^qX_2 + e^qX_0X_2^q + bX_1^{q+1} = 0, \tag{1}$$

where $e \in GF(q^2)$ and $b = -(e + e^q)$. Obviously, this equation represents a Hermitian curve if and only if

$$e \neq 0 \qquad \text{and} \qquad e^{q-1} \neq -1. \tag{2}$$

Now, let $\mathcal{H}_1$ be another Hermitian curve of $PG(2, q^2)$, containing the C_F-set $\mathcal{K}$ and different from $\mathcal{H}$. Let $\mathcal{H}_1$ be represented by the equation

$$e_1 X_0^q X_2 + e_1^q X_0 X_2^q - (e_1 + e_1^q) X_1^{q+1} = 0, \tag{3}$$

with $e_1 \in GF(q^2)^*$ and $e_1^{q-1} \neq -1$.

We study the intersection between $\mathcal{H}$ and $\mathcal{H}_1$. By (1), we get

$$X_1^{q+1} = \frac{e X_0^q X_2 + e^q X_0 X_2^q}{e + e^q}$$

and replacing in (3) we obtain

$$e_1 X_0^q X_2 + e_1^q X_0 X_2^q - \frac{e_1 + e_1^q}{e + e^q} (e X_0^q X_2 + e^q X_0 X_2^q) = 0 \Leftrightarrow$$
$$X_0^q X_2 \left(\frac{e_1 e^q - e e_1^q}{e + e^q} \right) + X_0 X_2^q \left(\frac{e e_1^q - e_1 e^q}{e + e^q} \right) = 0. \tag{4}$$

Set $\lambda := e_1 e^q - e e_1^q$ and assume $\lambda \neq 0$. Eq. (4) is equivalent to

$$\lambda X_0^q X_2 + \lambda^q X_0 X_2^q = 0 \Leftrightarrow \lambda X_0 X_2 (X_0^{q-1} - X_2^{q-1}) = 0. \tag{5}$$

If $X_0 = 0$ or $X_2 = 0$, we get the points $P_\infty = (0, 0, 1)$ and $P_0 = (1, 0, 0)$, respectively, which belong to $\mathcal{H} \cap \mathcal{H}_1$. If $X_0 \neq 0$ and $X_2 \neq 0$ the solutions of (5) are those of the equation $\alpha^{q-1} = 1$, where $\alpha = \frac{X_0}{X_2}$. The latter equation admits $q - 1$ distinct solutions $\alpha_1, \ldots, \alpha_{q-1}$ in $GF(q^2)$. For any $i \in \{1, \ldots, q - 1\}$ the intersection $\mathcal{H} \cap \mathcal{H}_1$ contains the points whose homogeneous coordinates are (X_0, X_1, X_2) where $X_2 \neq 0$, $X_0 = \alpha_i X_2$ and $X_1^{q+1} = \frac{e\alpha_i^q + e^q \alpha_i}{e + e^q} X_2^{q+1} = \alpha_i X_2^{q+1}$. Hence, for any $i \in \{1, \ldots, q - 1\}$, there are $q + 1$ such points. Thus, $|\mathcal{H} \cap \mathcal{H}_1| = (q - 1)(q + 1) + 2 = q^2 + 1$. Consequently, $\mathcal{H} \cap \mathcal{H}_1 = \mathcal{K}$.

If $\lambda = 0$, then $\mathcal{H} = \mathcal{H}_1$. But the condition $\lambda = 0$ is equivalent to the equation $ee_1^q - e^q e_1 = 0$, that is, $x - x^q = 0$, where $x = ee_1^q \in GF(q^2)^*$. The latter equation admits $q - 1$ distinct solutions. Therefore, by taking into account (1), (2) and this last remark, we are done. □

Remark 2.8. Notice that Kestenband, in [6], already observed that two Hermitian curves can intersect in $q^2 + 1$ points.

2.3. *A new set on* $\mathcal{H}(2, q^2)$ *of type* $[0, 1, 2, q + 1]_1$

In this section we construct a subset of $PG(2, q^2)$, q odd, of type $[0, 1, 2, q + 1]_1$ different from the set $\mathcal{K}$.

Let $\mathcal{H}(2, q^2)$ be the Hermitian curve of $PG(2, q^2)$ with equation

$$X_0 X_2^q + X_0^q X_2 - 2X_1^{q+1} = 0.$$

Consider the non-singular conic of $PG(2, q^2)$ with equation $X_0 X_2 - X_1^2 = 0$. Such a conic commutes with $\mathcal{H}(2, q^2)$. Denote by $\mathcal{C}$ the intersection between such a conic and $\mathcal{H}(2, q^2)$. We have already shown that $\mathcal{C}$ is a conic defined over $GF(q)$. The subconic $\mathcal{C}$ corresponds to the point set

$$\{(1, t, t^2) \mid t \in GF(q^2)^*, t^{q-1} = 1\} \cup \{(1, 0, 0), (0, 0, 1)\}.$$

Set $P := (0, 0, 1)$ and recall that the elements of the root group H' of $SU_3(q^2)$ with centre P and axis $P^\perp$ are represented by matrices of the form

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in GF(q^2)$ and $\alpha^q + \alpha = 0$. Such a group acts on $\mathcal{C}$ fixing only the point P. The action of H' takes $\mathcal{C}$ to the point set

$$\begin{aligned} \mathcal{F} = &\{(1, t, t^2 + \alpha) \mid t, \alpha \in GF(q^2),\ t^{q-1} = 1,\ \alpha^q + \alpha = 0\} \\ &\cup \{(1, 0, \alpha) \mid \alpha \in GF(q^2), \alpha + \alpha^q = 0\} \cup \{(0, 0, 1)\}. \end{aligned}$$

By definition, $|\mathcal{F}| = q^2 + 1$.

Proposition 2.9. *The set $\mathcal{F}$ is a subset of $\mathcal{H}(2, q^2)$ of type* $[0, 1, 2, q + 1]_1$.

Proof. Since $\mathcal{F} \subset \mathcal{H}(2, q^2)$ there exist 0-secants and 1-secants to $\mathcal{F}$. We must show that any other line of $PG(2, q^2)$ which is neither external nor tangent to $\mathcal{F}$ is i-secant to $\mathcal{F}$, $i \in \{2, q+1\}$. Set

$$\begin{aligned} &\mathcal{A} := \{(1, t, t^2 + \alpha) \mid t, \alpha \in GF(q^2),\ t^{q-1} = 1,\ \alpha^q + \alpha = 0\}, \\ &\mathcal{B} := \{(1, 0, \alpha) \mid \alpha \in GF(q^2), \alpha + \alpha^q = 0\}, \end{aligned}$$

then $\mathcal{F} = \mathcal{A} \cup \mathcal{B} \cup \{P\}$.

First of all, consider the line PP_α joining the point P with the point $P_\alpha := (1, 0, \alpha)$ of the set $\mathcal{B}$. Such a line corresponds to the point set

$$\{(1, 0, \alpha + \lambda) \mid \lambda \in GF(q^2)\} \cup \{(0, 0, 1)\}.$$

The line PP_α contains no point of the set $\mathcal{A}$, whereas it meets the set $\mathcal{B}$ in all of its points. It follows that such a line is $(q + 1)$-secant to $\mathcal{F}$.

Now, take the line $P_{\alpha_1} P_{\alpha_2}$ joining two distinct points $P_{\alpha_1} := (1, 0, \alpha_1)$ and $P_{\alpha_2} := (1, 0, \alpha_2)$ of $\mathcal{B}$. Such a line is the following point set

$$\{(1 + \lambda, 0, \alpha_1 + \lambda\alpha_2) \mid \lambda \in GF(q^2)\} \cup \{(1, 0, \alpha_2)\}.$$

It can be easily verified that the line $P_{\alpha_1} P_{\alpha_2}$ is 2-secant to $\mathcal{F}$. The same result is obtained when we consider both the line joining two distinct points of $\mathcal{A}$ and the line joining a point of $\mathcal{A}$ with a point of $\mathcal{B}$.

Finally, let PP_{t_α} be the line through the point P and the point $P_{t_\alpha} := (1, t, t^2 + \alpha)$ of $\mathcal{A}$. The line PP_{t_α} is the point set

$$\{(1, t, t^2 + \alpha + \lambda) \mid \lambda \in GF(q^2)\} \cup \{(0, 0, 1)\}.$$

It is straightforward to prove that no point of $\mathcal{B}$ lies on the line PP_{t_α}, whereas the intersection between PP_{t_α} and $\mathcal{A}$ is the following point set

$$\{(1, t, t^2 + \alpha) \mid \alpha \in GF(q^2),\ \alpha + \alpha^q = 0\} \cup \{(0, 0, 1)\}.$$

Hence, the line PP_{t_α} is $(q + 1)$-secant to $\mathcal{F}$. From the previous arguments it also follows that there are q $(q + 1)$-secant lines to $\mathcal{F}$ which pass through the point P. $\square$

3. Some geometry of commuting symplectic and unitary polarities

3.1. C_F-sets and symplectic spreads

Our main purpose here is to prove the following theorem.

Theorem 3.1. *To a C_F-set of $\mathcal{H}(2, q^2)$ there corresponds a regular symplectic spread of $PG(3, q)$, and conversely.*

Let X_0, X_1, X_2, X_3 be homogeneous projective coordinates in $PG(3, q^2)$, q even. Let $\omega \in GF(q^2) \setminus GF(q)$ and $\mathcal{H}$ be the Hermitian surface of $PG(3, q^2)$ with equation

$$\omega X_0 X_2^q + X_1 X_3^q + \omega^q X_2 X_0^q + X_3 X_1^q = 0$$

and let π be the plane $X_3 = \omega^q X_1$. The plane π is secant to $\mathcal{H}$ and let $\bar{\mathcal{H}} := \mathcal{H} \cap \pi$ be the Hermitian curve with equation

$$\omega X_0 X_2^q + (\omega + \omega^q) X_1^{q+1} + \omega^q X_2 X_0^q = 0\,.$$

It is easy to see that the set

$$\Sigma_0 := \{(\omega^{-1}\rho a^q, b, \rho b^q, a) \mid a, b \in GF(q^2),\ \rho^{q+1} = 1\}$$

is a symplectic subgeometry $\mathcal{W}_3(q)$ embedded in $\mathcal{H}$ (see [8]). Next, consider the family of q^2+1 lines of $PG(3, q^2)$ given by the equations

$$r_t : \begin{cases} X_1 = t X_0 \\ X_2 = \omega^{-q} t^q X_3 \end{cases} \qquad t \in GF(q^2) \cup \{\infty\}. \tag{6}$$

It turns out that all these lines are generators of $\mathcal{H}$.

We show that the lines r_t's are pairwise skew.

Obviously, $r_\infty \cap r_t = \emptyset$ for any $t \in GF(q^2)$. The two lines r_{t_1} and r_{t_2}, with $t_1, t_2 \in GF(q^2)$, are skew if and only if

$$\begin{vmatrix} t_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega^{-q} t_1^q \\ t_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega^{-q} t_2^q \end{vmatrix} \neq 0,$$

if and only if $t_1 \neq t_2$. Therefore, these lines form a (q^2+1)-span of $\mathcal{H}$.

Each line r_t meets the plane π at the point P_t. Since

$$r_t \cap \pi : \begin{cases} X_1 = t X_0 \\ X_2 = \omega^{-q} t^q X_3 \\ X_3 = \omega^q X_1 \end{cases} \Leftrightarrow \begin{cases} X_1 = t X_0 \\ X_2 = t^q X_1 \\ X_3 = \omega^q X_1 \end{cases}$$

it is straightforward that $P_\infty = (0, 0, 1)$ and $P_t = (1, t, t^{q+1})$, for any $t \in GF(q^2)$.

Set

$$\mathcal{L} := \{(1, t, t^{q+1}, \omega^q t) \mid t \in GF(q^2)\} \cup \{(0, 0, 1, 0)\}$$

and

$$\mathcal{K} := \{(1, t, t^{q+1}) \mid t \in GF(q^2)\} \cup \{(0, 0, 1)\}.$$

Therefore, the set $\mathcal{K}$ coincides with the C_F-set studied in the previous section.

Now, we need the following result.

Proposition 3.2. $\mathcal{K} \cap \Sigma_0 = \{(1, t, 1) \mid t \in GF(q^2)^*, t^{q+1} = 1\}$.

Proof. It is clear that P_0 and P_∞ do not lie on the symplectic subgeometry Σ_0. Let $P_t = (1, t, t^{q+1}, \omega^q t)$ be a point of $\mathcal{L}$ with $t \in GF(q^2)^*$. Then

$$P_t \in \Sigma_0 \Leftrightarrow \begin{cases} \omega^{-1}\rho a^q = 1 \\ b = t \\ \rho b^q = t^{q+1} \\ a = \omega^q t. \end{cases} \Leftrightarrow \rho t^q = t^{q+1} \Leftrightarrow t = \rho$$

Since ρ is a $(q + 1)$-st root of unity in $GF(q^2)$, $|\mathcal{K} \cap \Sigma_0| = q + 1$. $\square$

Remark 3.3. All $q + 1$ points of $\mathcal{K} \cap \Sigma_0$ are collinear. This fact can be seen by direct computations. It is also a consequence of [10].

Now, for each point $P_t \in \mathcal{L}$ the equation of the tangent plane to $\mathcal{H}$ at P_t is the following:

$$\pi_{P_t} : \omega t^{q+1} X_0 + \omega t^q X_1 + \omega^q X_2 + t^q X_3 = 0.$$

In particular,

$$\pi_{P_0} : X_2 = 0 \qquad \text{and} \qquad \pi_{P_\infty} : X_0 = 0.$$

Proposition 3.4. *Any line r_t, with $t \in GF(q^2) \cup \{\infty\}$, meets the symplectic subgeometry Σ_0 in a Baer subline. Such sublines form a symplectic spread of Σ_0.*

Proof. Firstly consider the intersection between r_∞ and Σ_0. A point of r_∞ has homogeneous projective coordinates $(0, X_1, X_2, 0)$. By definition $P = (0, 0, 1, 0) \notin \Sigma_0$ and $P_\lambda = (0, 1, \lambda, 0)$, where $\lambda = \frac{X_2}{X_1}$ and $X_1 \neq 0$, is a point of Σ_0 if and only if

$$\begin{cases} b = 1 \\ \rho b^q = \lambda \end{cases} \Leftrightarrow \lambda = \rho.$$

It follows that $r_\infty \cap \Sigma_0 = \{(0, 1, \lambda, 0) \mid \lambda^{q+1} = \rho^{q+1} = 1\}$, and so $|r_\infty \cap \Sigma_0| = q + 1$.

Next, we investigate the intersection between r_0 and Σ_0. A point which lies on r_0 has homogeneous projective coordinates $(X_0, 0, 0, X_3)$. Obviously, $P = (0, 0, 0, 1) \notin \Sigma_0$. Further, a point $P_\lambda = (1, 0, 0, \lambda)$, with $\lambda = \frac{X_3}{X_0}$ and $X_0 \neq 0$, lies on the subgeometry Σ_0 if and only if

$$\begin{cases} \omega^{-1}\rho a^q = 1 \\ a = \lambda, \end{cases}$$

hence $\lambda^q = \frac{\omega}{\rho}$. This equation admits a unique solution $\lambda = \frac{\omega^q}{\rho^q} = \omega^q \rho$. As above, it follows that $|r_0 \cap \Sigma_0| = q + 1$.

Finally, assuming $t \in GF(q^2)^*$, we observe that a point of r_t has homogeneous projective coordinates $(X_0, tX_0, \omega^{-q} t^q X_3, X_3)$. It is clear that $P = (0, 0, \omega^{-q} t^q, 1) \notin \Sigma_0$, whereas a point $P_\lambda = (1, t, \omega^{-q} t^q \lambda, \lambda)$, with $\lambda = \frac{X_3}{X_0}$ and $X_0 \neq 0$, lies on the subgeometry Σ_0 if and only if

$$\begin{cases} \omega^{-1}\rho a^q = 1 \\ b = t \\ \rho b^q = \omega^{-q} t^q \lambda \\ a = \lambda, \end{cases}$$

hence $\lambda = \omega^q \rho$. Arguing as in the previous cases, $|r_t \cap \Sigma_0| = q + 1$.

From [10, p. 379] it follows that for any point $P_t \in \mathcal{L}$ we can choose a Baer subline $r_t \cap \Sigma_0$. More precisely, if $P_t \in \Sigma_0$ then the tangent plane π_{P_t} intersects Σ_0 in a Baer subplane containing the Baer subline $r_t \cap \Sigma_0$; otherwise π_{P_t} meets Σ_0 exactly in $r_t \cap \Sigma_0$, which turns out to be a Baer subline. Since the q^2+1 sublines $r_t \cap \Sigma_0$, $t \in GF(q^2) \cup \{\infty\}$, are mutually skew, they form a spread of Σ_0. Each subline is totally isotropic with respect to the symplectic polarity associated with Σ_0 (see [8]). Therefore, the second part of the statement follows. □

Now, we prove a more general result.

Proposition 3.5. *For any pair $\{l, m\}$ of skew generators of a Hermitian surface $\mathcal{H}(3, q^2)$, there is an element of $PGU_4(q^2)$ that maps $\{l, m\}$ to a pair of skew conjugate generators of $\mathcal{H}(3, q^2)$.*

Proof. First of all, if r is a line of $PG(3, q^2)$ we denote by r^q the line containing the points of the form $x^q = (c^q, d^q, a^q, b^q)$ as $x = (a, b, c, d)$ runs over the points of the line r. The lines r and r^q are said to be *conjugate*.

We begin by showing the following property.

(∗) If l is a generator of $\mathcal{H}(3, q^2)$, then the stabilizer of l in $PGU_4(q^2)$ is transitive on the lines of $\mathcal{H}(3, q^2)$ skew to l.

By Witt's theorem $PGU_4(q^2)$ acts transitively on generators of $\mathcal{H}(3, q^2)$. Let V be the 4-dimensional vector space underlying $PG(3, q^2)$, f the non-degenerate Hermitian form fixed by $PGU_4(q^2)$, and $W \subset V$ the totally isotropic 2-dimensional vector subspace underlying l. Let $\{w_1, w_2\}$ be a basis for W. By [11], there exists a 2-dimensional totally singular subspace U of V with basis $\{u_1, u_2\}$ such that $V = W \oplus U$ and $f(w_i, u_i) = 1$ for $i = 1, 2$ and $f(w_i, u_j) = 0$ for $i \neq j$. We denote by G the stabilizer of W in $U_4(q^2)$. With respect to the basis $\{u_1, u_2, w_1, w_2\}$, the elements of G are represented by matrices of the form

$$\begin{pmatrix} X & O_2 \\ Y & Z \end{pmatrix},$$

where O_2, X, Y, Z are 2×2 matrices over $GF(q^2)$, such that O_2 is the null matrix, ${}^tX\, Z^q = I_2$, ${}^tY\, X^q + {}^tX\, Y^q = O_2$, $\det(XZ) \neq 0$. The set of elements of G for which $Y + {}^tY^q = O_2$ forms an elementary abelian subgroup E of order q^4 acting transitively on generators of $\mathcal{H}(3, q^2)$ skew to l, proving (∗).

For the sake of simplicity, we now suppose that $\mathcal{H}(3, q^2)$ has canonical equation

$$X_0 X_2^q + X_1 X_3^q + X_2 X_0^q + X_3 X_1^q = 0. \tag{7}$$

From (7), it follows that if r is a generator of $\mathcal{H}(3, q^2)$, so is r^q. It should be noted that $\mathcal{H}(3, q^2)$ has a skew conjugate pair of generators; namely, the line r with equations $X_0 = X_1 = 0$ and the line r^q defined as above. By Witt's theorem $PGU_4(q^2)$ acts transitively on the generators of $\mathcal{H}(3, q^2)$, and hence there exists an element $g \in PGU_4(q^2)$ such that $l^g = r$. Obviously m^g and r are skew. Therefore m^g and r^q are both skew to r. Denote by L the stabilizer of r in $PGU_4(q^2)$. By property (∗) there exists $h \in L$ such that $m^{gh} = r^q$. Thus, gh is the desired element of $PGU_4(q^2)$. □

The previous proposition allows us to show that the symplectic spread formed by the Baer sublines $r_t \cap \Sigma_0$, $t \in GF(q^2) \cup \{\infty\}$, is regular. Indeed, all such lines meet the two skew generators l and m of $\mathcal{H}$ with equations

$$l : \begin{cases} X_0 = 0 \\ X_1 = 0, \end{cases} \qquad m : \begin{cases} X_2 = 0 \\ X_3 = 0. \end{cases}$$

By Proposition 3.5, it is always possible to assume $m = l^q$. Therefore, this symplectic spread is regular (see [5, p. 201]).

Conversely, given a regular symplectic spread $\mathcal{S}$ of Σ_0, after extending over $GF(q^2)$ its members, and intersecting with a Hermitian curve obtained by sectioning $\mathcal{H}(3, q^2)$ with a secant plane one can see that the resulting set is actually a C_F-set. This depends on the fact that $PSp_4(q)$ acts transitively on regular symplectic spreads of Σ_0 and so we can always reduce to the case when $\mathcal{S}$ is the spread constructed above.

The main theorem has been completely proved.

Remark 3.6. Let λ be a $(q-1)$-st root of unity in $GF(q^2)$. Then, by Remark 2.3, r_λ : $X_0 + \lambda X_2 = 0$ is a $(q+1)$-secant line to $\mathcal{K}$ and it turns out that

$$r_\lambda \cap \mathcal{K} = \left\{(1, t, t^{q+1}) \mid t \in GF(q^2)^*, t^{q+1} = -\frac{1}{\lambda}\right\}.$$

By Proposition 3.4, to any point $P_t := (1, t, t^{q+1})$ of $r_\lambda \cap \mathcal{K}$ there corresponds the Baer subline

$$r_t \cap \Sigma_0 : \begin{cases} X_1 = tX_0 \\ X_2 = \omega^{-q} t^q X_3. \end{cases}$$

Since $\lambda^{q-1} = 1$, the equation $t^{q+1} = -\frac{1}{\lambda}$ admits $q+1$ distinct solutions in $GF(q^2)$. Hence, with the points of $r_\lambda \cap \mathcal{K}$ one can associate $q+1$ Baer sublines. By Proposition 3.4, these sublines, belonging to a symplectic spread of Σ_0, are pairwise skew and form a regulus of the hyperbolic quadric $\mathcal{Q}^+(3, q)$ of $PG(3, q)$ with equation $X_1X_2 + \frac{1}{\lambda}\omega^{-q} X_0X_3 = 0$.

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